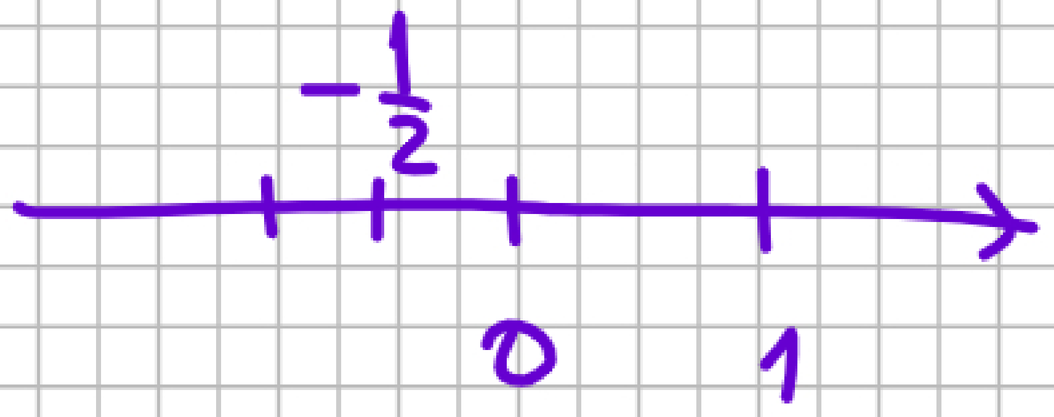
1. Calculati integrala dubla

$$\iint_D 4xy + 2x dx dy$$

unde domeniul D este marginit de

$y = x^2 + x, y = 2x^3, x \geq 0$

$$\iint_D -xy + 2x \, dx \, dy$$



$$\begin{cases} y = x^2 + x \\ y = 2x^3 \end{cases} \Leftrightarrow 2x^3 = x^2 + x \Leftrightarrow x(2x^2 - x - 1) = 0 \Rightarrow x_1 = 0$$

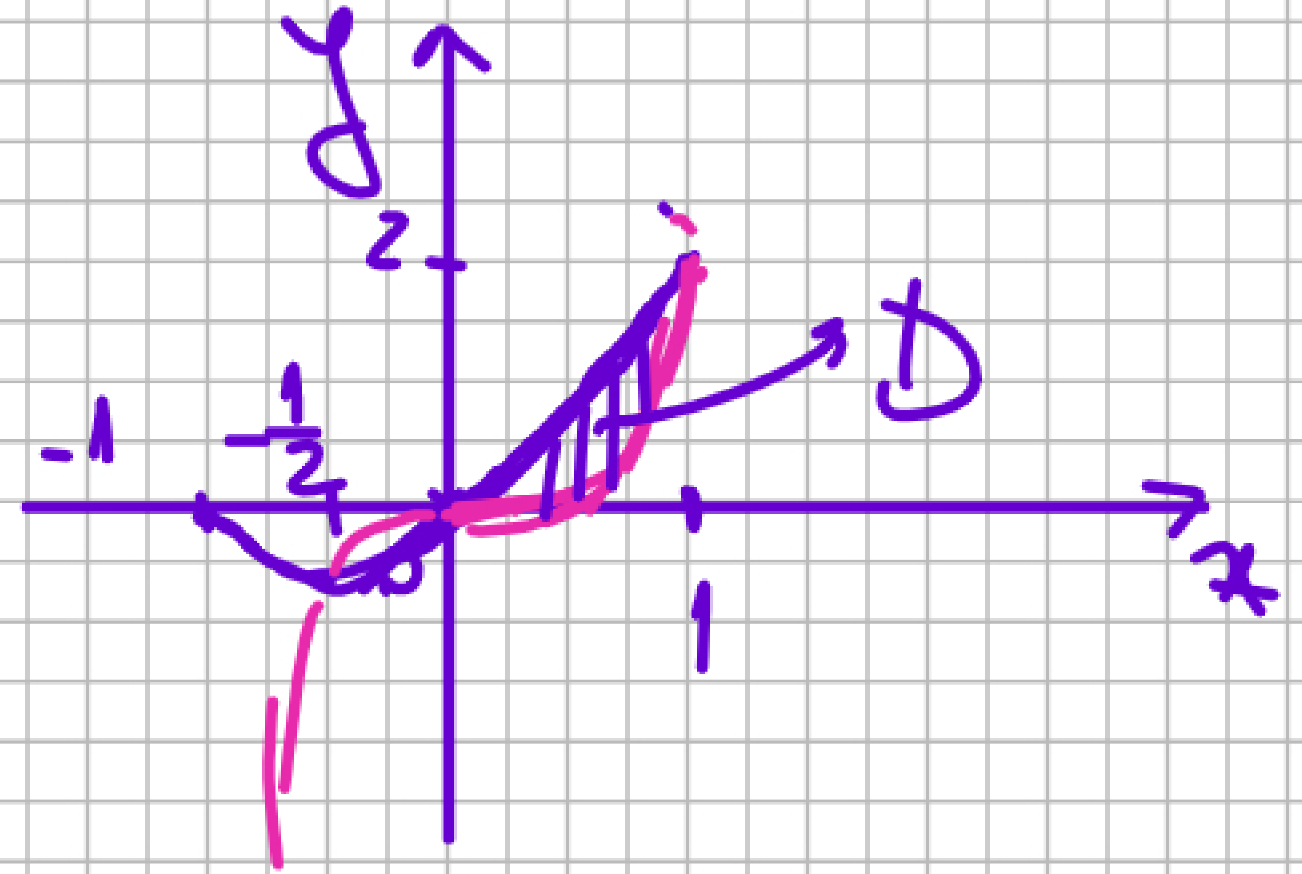
$$\Rightarrow 2x^2 - x - 1 = 0$$

$$\Delta = 1 + 4 \cdot 2 = 9, \sqrt{\Delta} = 3$$

$$x_{2/3} = \frac{1 \pm 3}{4} \begin{cases} 1 \\ -\frac{1}{2} \end{cases}$$

$$\Rightarrow x \in [0, 1]$$

$$y \in [2x^3, x^2 + x]$$



$$D=\{(x,y)\in\mathbb{R}^2 \,/\, 0\le x\le 1,\ 2x^3\le y\le x^2+x\}$$

$$\iint_D -xy+2x\,dx\,dy=\int_0^1\left(\int_{2x^3}^{x^2+x}-xy+2x\,dy\right)dx=\int_0^1\left(2x^7-\frac{1}{2}x^5-5x^4+\frac{3}{2}x^3+2x^2\right)dx$$

$$=\left(\frac{\cancel{2}x^8}{\cancel{8}_4}-\frac{1}{2}\cdot\frac{x^6}{6}-x^5+\frac{3}{2}\cdot\frac{x^4}{4}+2\cdot\frac{x^3}{3}\right)\Big|_0^1=$$

$$\int_{2x^3}^{x^2+x}(-xy+2x)\,dy=-x\cdot\frac{y^2}{2}\Big|_{2x^3}^{x^2+x}+2xy\Big|_{2x^3}^{x^2+x}=-\frac{x}{2}\left[(x^2+x)^2-(2x^3)^2\right]$$

$$+2x(x^2+x-2x^3)=-\frac{x}{2}\left[x^4+2x^3+x^2-4x^6\right]+2x^3+2x^2-4x^4=$$

$$=-\frac{1}{2}x^5-x^4-\frac{1}{2}x^3+2x^7+2x^3+2x^2-4x^4=2x^7-\frac{1}{2}x^5-5x^4+\frac{3}{2}x^3+2x^2$$

$$= {}^{2)}\frac{1}{4} - \frac{1}{12} - 1 + {}^{4)}\frac{3}{8} + {}^{3)}\frac{2}{3} = {}^{2)}\frac{5}{8} + {}^{24)}\frac{7}{12} - 1 = \frac{15+14-24}{24} = \frac{29-24}{24} = \frac{5}{24}$$

2. Derivata Frechet a functiei

$$f(x, y) = x^3 + y^3 - 3xy$$

in punctul

$$a = (-1, 2)$$

$$d_a f : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$d_a f(h) = \frac{\partial f(a)}{\partial x} \cdot h_1 + \frac{\partial f(a)}{\partial y} \cdot h_2 = -3h_1 + 15h_2$$

$$\frac{\partial f}{\partial x} = 3x^2 - 3y \Rightarrow \frac{\partial f(a)}{\partial x} = 3\cdot(-1)^2 - 3\cdot 2 = 3 - 6 = -3$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3x \Rightarrow \frac{\partial f(a)}{\partial y} = 3\cdot 2^2 - 3\cdot(-1) = 12 + 3 = 15$$

4. Derivata

$\nabla_u f(a)$

a functiei

$f(x, y, z) = z^4 \sqrt{3} \cos(3x + y^3)$

dupa directia

$u = \dfrac{v}{||v||}$ unde $v = (-3, -3, 3)$

in punctul

$a = (9, -3, -1)$

$$\nabla_u f(a) = \nabla f(a) \cdot u$$

$$\nabla f(a) = \left(\frac{\partial f(a)}{\partial x}, \frac{\partial f(a)}{\partial y}, \frac{\partial f(a)}{\partial z} \right)$$

$$||v|| = \sqrt{(-3)^2 + (-3)^2 + 3^2} = \sqrt{3 \cdot 9} = 3\sqrt{3}$$

$$u = \frac{1}{3\sqrt{3}} \cdot v = \left(-\frac{3}{3\sqrt{3}}, -\frac{3}{3\sqrt{3}}, \frac{3}{3\sqrt{3}} \right) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$$\frac{\partial f}{\partial x} = z^4 \cdot \sqrt{3} \cdot (-\sin(3x + y^3)) \cdot 3 = -3\sqrt{3}\, z^4 \sin(3x + y^3)$$

$$\frac{\partial f}{\partial y} = z^4 \sqrt{3} \cdot (-\sin(3x + y^3)) \cdot 3y^2 = -3\sqrt{3}\, y^2 z^4 \sin(3x + y^3)$$

$$\frac{\partial f}{\partial z} = 4z^3 \sqrt{3} \cos(3x + y^3)$$

$$3x + y^3 = 27 + (-3)^3 = 0$$

$$\frac{\partial f(a)}{\partial x} = \frac{\partial f(a)}{\partial y} = 0$$

$$\frac{\partial f(a)}{\partial z} = 4 \cdot (-1)^3 \cdot \sqrt{3}\,\underbrace{\cos(0)}_{1} = -4\sqrt{3}$$

$$\nabla f(a) = (0, 0, -4\sqrt{3})$$

$$\nabla_u f(a) = (0, 0, -4\sqrt{3}) \cdot \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = 0 \cdot \left(-\frac{1}{\sqrt{3}}\right) + 0\left(-\frac{1}{\sqrt{3}}\right) + (-4\cancel{\sqrt{3}}) \cdot \frac{1}{\cancel{\sqrt{3}}} =$$

$$= -4$$

$$\int_0^{\frac{\pi}{2}}\left(\int_0^6 y\sin(x) + y^2\cos(x)\, dy\right)dx = \int_0^{\frac{\pi}{2}}\left(\sin(x)\cdot\frac{y^2}{2} + \cos(x)\cdot\frac{y^3}{3}\right)\Big|_0^6 dx$$

$$= \int_0^{\frac{\pi}{2}} 18\sin x + 72\cos(x)\, dx = -18\cos(x)\Big|_0^{\frac{\pi}{2}} + 72\sin(x)\Big|_0^{\frac{\pi}{2}} = 18 + 72 = 90.$$

7. Coeficientii Fourier

$b_n, n \geq 1$

ai functiei

$f(x) = x + |x| = \begin{cases} 2x, & x \geq 0 \\ 0, & x < 0 \end{cases}$

definita pe intervalul

$[-\pi, \pi]$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx, \ (\forall)\, n \geq 1$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx), \ (\forall)\, n \geq 1$$

$$f(-x) = -x + |-x| = -x + |x|$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 2x\,dx = \frac{1}{\pi}\cdot x^2\Big|_0^{\pi} = \frac{\pi^2}{\pi} = \pi$$

$\sin(n\pi) = 0$

$\cos(n\pi) = (-1)^n$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_0^{\pi} 2x\cdot\cos(nx)\,dx = \frac{2}{\pi}\left[\underbrace{\frac{1}{n}x\sin(nx)}_{0}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right] =$$

$f(x) = x \Rightarrow f'(x) = 1$

$g'(x) = \cos(nx)$ $\quad g(x) = \frac{\sin(nx)}{n}$

$$= -\frac{2}{n\pi}\int_0^{\pi}\sin(nx)\,dx = -\frac{2}{n\pi}\cdot\frac{-\cos(nx)}{n}\Big|_0^{\pi} = \frac{2}{n^2\pi}\left(\cos(n\pi) - 1\right) = \begin{cases} 0, & n \text{ par} \\ -\frac{4}{n^2\pi}, & n \text{ impar}\end{cases}$$

$$\frac{2\left((-1)^n - 1\right)}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{0}^{\pi} 2x\sin(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} x\cdot\left(\frac{-\cos(nx)}{n}\right)' dx =$$

$$= \frac{2}{\pi}\left(-\frac{1}{n}x\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right) =$$

$$= \frac{2}{\pi}\left(-\frac{1}{n}\pi\cdot\underbrace{\cos(n\pi)}_{(-1)^n} + \frac{1}{n}\cdot\underbrace{\frac{\sin(nx)}{n}\Big|_0^{\pi}}_{0}\right) =$$

$$= -\frac{2}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{n}\cdot(-1)^n = -\frac{2}{n}\cdot(-1)^n$$

Serie Fourier: $\dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$

10. Limita functiei

$$f(x,y) = \frac{xy^2}{x^2+y^4}$$

in punctul

$(0,0)$

$y=0: f(x,0) = \frac{x\cdot 0}{x^2+0} = \frac{0}{x^2} = 0 \xrightarrow[x\to 0]{} 0$

$y=\sqrt{x}: f(x,\sqrt{x}) = \frac{x\cdot\sqrt{x}^2}{x^2+\sqrt{x}^4} = \frac{x^2}{2x^2} = \frac{1}{2} \xrightarrow[x\to 0]{} \frac{1}{2}$

$\Rightarrow$

$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$

9. Punctele critice ale functiei

$$f(x,y) = 2x^3 + 6xy^2 - 60x - 36y + 6$$

se clasifica astfel (alegeti toate raspunsurile corecte):

$\begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$

$\frac{\partial f}{\partial x} = 6x^2 + 6y^2 - 60$

$\frac{\partial f}{\partial y} = 12xy - 36$

$\begin{cases} 6x^2+6y^2 = 60 \mid :6 \\ 12xy = 36 \mid :12 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} x^2+y^2 = 10 \\ xy = 3 \mid \cdot 2 \end{cases}$

$$\begin{cases} x^2+y^2=10 \\ xy=3 \end{cases} \Rightarrow x \text{ și } y \text{ au aceloși semn}$$

$$(x+y)^2 = x^2+y^2+2xy = 10+6=16 \Rightarrow x+y=\pm 4$$

- $\begin{cases} x+y=4 \\ xy=3 \end{cases}$ $\quad t^2-4t+3=0 \Leftrightarrow (t-1)(t-3)=0$ $\left\langle \begin{array}{l} t_1=1 \\ t_2=3 \end{array} \right.$

$$\Rightarrow \begin{cases} x=1 \\ y=3 \end{cases} \text{ sau } \begin{cases} x=3 \\ y=1 \end{cases}$$

- $\begin{cases} x+y=-4 \\ xy=3 \end{cases}$ $\quad t^2+4t+3=0 \Leftrightarrow (t+1)(t+3)=0$ $\left\langle \begin{array}{l} t_3=-1 \\ t_4=-3 \end{array} \right.$

$$\Rightarrow \begin{cases} x=-1 \\ y=-3 \end{cases} \text{ sau } \begin{cases} x=-3 \\ y=-1 \end{cases}$$

$$\left| \begin{array}{l} \dfrac{\partial f}{\partial x} = 6x^2+6y^2-60 \\ \dfrac{\partial f}{\partial y} = 12xy-36 \end{array} \right.$$

Punctele critice sunt $(1,3), (3,1), (-1,-3), (-3,-1)$.

$$\frac{\partial^2 f(x)}{\partial x^2} = 12x, \quad \frac{\partial^2 f}{\partial x \partial y} = 12y, \quad \frac{\partial^2 f}{\partial y \partial x} = 12y, \quad \frac{\partial^2 f}{\partial y^2} = 12x$$

Matricea hessiană: $H_{(x,y)} f = \begin{bmatrix} 12x & 12y \\ 12y & 12x \end{bmatrix}$

$$H_{(1,3)} f = \begin{bmatrix} 12 & 36 \\ 36 & 12 \end{bmatrix}; \quad \begin{array}{l} \Delta_1 = 12 > 0 \\ \Delta_2 = 12^2 - 36^2 < 0 \end{array} \Rightarrow (1,3) \text{ punct șa}$$

$H_{(3,1)}f = \begin{bmatrix} 36 & 12 \\ 12 & 36 \end{bmatrix}$; $\Delta_1 = 36 > 0$

$\Delta_2 = 36^2 - 12^2 = 24 \cdot 48 > 0$ $\Rightarrow$ $(3,1)$ punct de minim local

$H_{(-1,-3)}f = \begin{bmatrix} -12 & -36 \\ -36 & -12 \end{bmatrix}$; $\Delta_1 = -12 < 0$

$\Delta_2 = 12^2 - 36^2 < 0$ $\Rightarrow$ $(-1,-3)$ punct șa

$H_{(-3,-1)}f = \begin{bmatrix} -36 & -12 \\ -12 & -36 \end{bmatrix}$; $\Delta_1 = -36 < 0$

$\Delta_2 = 36^2 - 12^2 > 0$ $\Rightarrow$ $(-3,-1)$ punct de maxim local

8. Pentru functia definita prin

$$f(x,y) = \frac{x^3 - y^3}{x^2 + y^2} \quad \text{daca} \quad (x,y) \neq q(0,0)$$

si

$$f(0,0) = 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left(\underbrace{\underbrace{\frac{x^2}{x^2+y^2}}_{\text{mărg.}} \cdot \underbrace{x}_{\to 0}}_{\to 0} - \underbrace{\underbrace{\frac{y^2}{x^2+y^2}}_{\text{mărg.}} \cdot \underbrace{y}_{\to 0}}_{\to 0} \right) = 0 - 0 = 0 = f(0,0)$$

$$\Rightarrow f \in C_{\{(0,0)\}}$$

Derivatele partiale în $(0,0)$:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t\to 0} \frac{f(t,0) - f(0,0)}{t - 0} = \lim_{t\to 0} \frac{\frac{t^3}{t^2} - 0}{t} = \lim_{t\to 0} \frac{t}{t} = \lim_{t\to 0} 1 = 1$$

$$\frac{\partial f(0,0)}{\partial y} = \lim_{t\to 0} \frac{f(0,t) - f(0,0)}{t} = \lim_{t\to 0} \frac{-\frac{t^3}{t^2} - 0}{t} = \lim_{t\to 0} \frac{-t}{t} = \lim_{t\to 0} (-1) = -1$$

$$\Rightarrow f \in \mathcal{D}_{\{(0,0)\}}$$

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2) - \overbrace{f(0,0)}^{0} - \nabla f(0,0)\cdot(h_1,h_2)}{\|h\|} = \lim_{(h_1,h_2)\to(0,0)} \frac{\frac{h_1^3 - h_2^3}{h_1^2+h_2^2} - \frac{h_1^2+h_2^2}{h_1^2+h_2^2}(h_1 - h_2)}{\sqrt{h_1^2+h_2^2}}$$

$$h = (h_1, h_2)$$

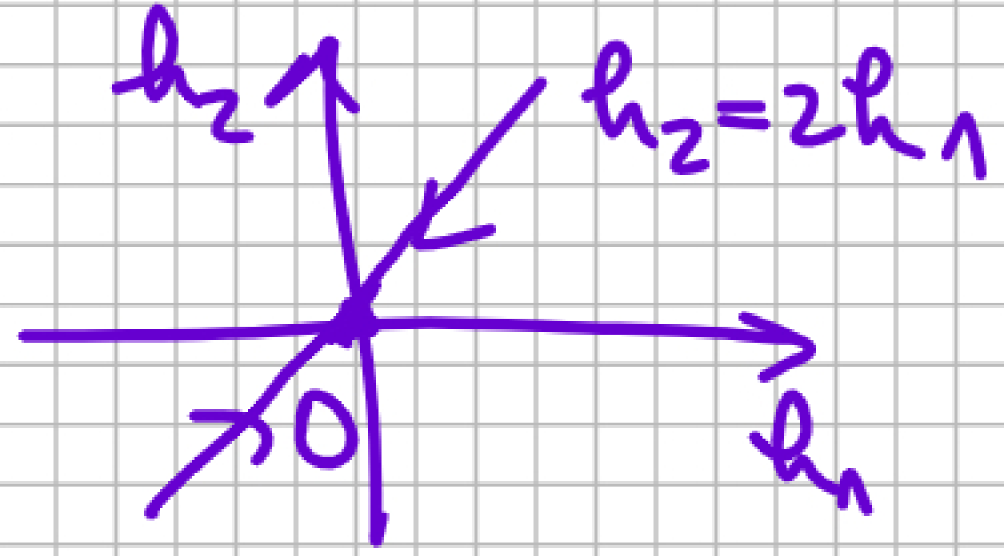
$$\nabla f(0,0) = \left(\frac{\partial f(0,0)}{\partial x}, \frac{\partial f(0,0)}{\partial y}\right) = (1,-1)$$

$$(1,-1)\cdot(h_1,h_2) = h_1 - h_2$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{h_1^2 h_2 - h_1 h_2^2}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}}$$

$$\sqrt{h_1^2+h_2^2} \sim \sqrt{h_1^2} = |h_1| = h_1$$

$$g(h_1,h_2)=\frac{h_1^2h_2-h_1h_2^2}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}}$$



$$h_2=2h_1: f(h_1,2h_1)=\frac{h_1^2\cdot 2h_1-h_1\cdot 4h_1^2}{2h_1^2\cdot\sqrt{2h_1^2}}=\frac{-2h_1^3}{2h_1^2\cdot\sqrt{2}|h_1|}$$

$$\left.\begin{aligned}
&\lim_{\substack{h_1\to 0\\ h_1>0}} f(h_1,2h_1)=\lim_{\substack{h_1\to 0\\ h_1>0}}\frac{-2h_1^3}{2\sqrt{2}h_1^3}=-\frac{1}{\sqrt{2}}\\
&\lim_{\substack{h_1\to 0\\ h_1<0}} f(h_1,2h_1)=\lim_{\substack{h_1\to 0\\ h_1<0}}\frac{-2h_1^3}{-2\sqrt{2}h_1^3}=\frac{1}{\sqrt{2}}
\end{aligned}\right\}\Rightarrow (\nexists)\lim_{(h_1,h_2)\to(0,0)} f(h_1,h_2)$$

$\Rightarrow f \notin \mathcal{F}_{\{(0,0)\}}$ (f nu este dif. Fréchet în (0,0))

$\Rightarrow$ f nu are deriv. parțiale continue în (0,0)

(dacă f are deriv. parțiale cont. în a, atunci f este dif. Fréchet în a)

$$\frac{\partial f}{\partial x}(x,y) = \left(\frac{x^3-y^3}{x^2+y^2}\right)'_x = \frac{3x^2(x^2+y^2)-(x^3-y^3)\cdot 2x}{(x^2+y^2)^2} = \frac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \dots$$

$$\frac{\partial f}{\partial x} = \begin{cases} \dfrac{x^4+3x^2y^2+2xy^3}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

$\frac{\partial f}{\partial x}$ - cont. în (0,0)